JAMES RUSE A H S

GESTA NON VERBA

YEAR 11
TERM 4 [YEARLY] EXAMINATION 2007

MATHEMATICS EXTENSION 1

Time Allowed – 90 Minutes
(Plus 5 minutes Reading Time)

All questions may be attempted

All questions are of equal value

Department of Education approved calculators and templates are permitted

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc.
Each question must show (in the top right hand corner) your Candidate Number.

QUESTION 1 **Marks**

		Marks
(a)	Find the exact value of $\cos\frac{7\pi}{6}$.	1
(b)	Evaluate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$.	3
(c)	Find all solutions to $\frac{x}{x^2-1} \geq 0$.	3
(d)	Use two applications of Simpson's Rule to evaluate $\int_1^5 \frac{\ln x}{x} dx$. (Answer to 1 decimal place.)	2

QUESTION 2 (9 marks)

(a)	Find the area of the region enclosed between the curves $y = 4x - x^2$ and $y = x^2 - 2x$.	3
(b)	Differentiate $y = \frac{1}{\cos^{-1} x}$.	2
(c)	A closed cylinder of radius R and height H has a fixed surface area of $2\pi N$ square units. Prove that its volume will be a maximum when $2R = H$.	4

QUESTION 3 (9 marks)

(a)	Use the substitution $u = 25 - x^2$ to evaluate $\int_3^4 \frac{2x}{\sqrt{25-x^2}} dx$.	3
(b)	(i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$, for $R > 0$ and α acute.	2
	(ii) Find the first positive value of x for which $\sqrt{3}\cos x - \sin x$ will be a maximum.	1
(c)	Find the equation of the tangent to the curve $y = \sqrt{9-(x-1)^2}$. if the tangent passes through the point (6,0).	3

QUESTION 4 (9 marks)

(a) (i) Differentiate $y = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)$. **2**

(ii) Hence,or otherwise, neatly sketch $y = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)$. **2**

(b) Find the equation of the tangent to the curve $y = 3\cos^{-1}\dfrac{x}{4}$ at the point where it crosses the $y-$ axis. **3**

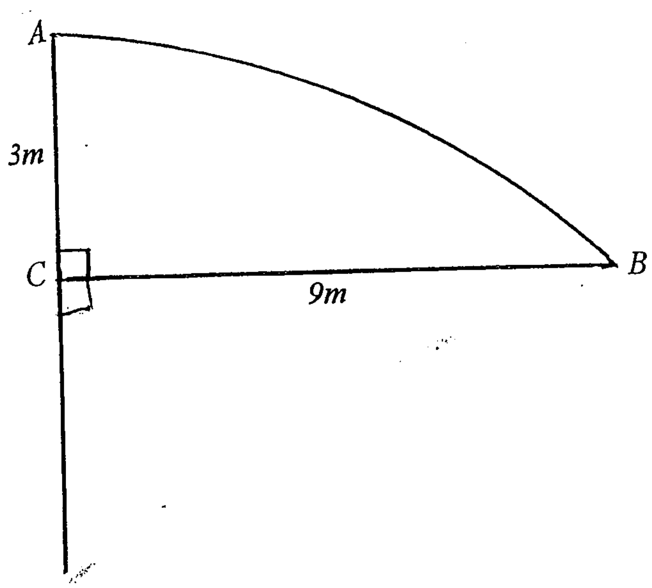
(c) Draw a neat sketch of $y = \dfrac{\pi}{2} + \sin^{-1} 2x$. **2**

QUESTION 5 (9 marks)

(a) In the shape below, $\angle ACB = 90^\circ$, $AC = 3$ metres, $BC = 9$ metres and AB is an arc of a circle whose centre is on the line passing through AC.

(i) Find the length of arc AB. (to 2 decimal places). **3**

(ii) Find the area of the shape ABC. (to 2 decimal places). **2**



(b) (i) Prove that $a^2 + b^2 \geq 2ab$ for all real values of a and b. **1**

(ii) Hence, or otherwise, prove that $(a+b)\left(\dfrac{1}{a} + \dfrac{1}{b}\right) \geq 4$ for $a > 0$ and $b > 0$. **3**

QUESTION 6 (9 marks)

(a) Prove that $\sin(2\sin^{-1}x + \cos^{-1}x) = \sqrt{1-x^2}$. **3**

(b) A function is defined by $f(x) = \dfrac{\sqrt{x^2-1}}{x}$.

(i) Find the largest positive domain for which $y = f(x)$ has an inverse. **1**

(ii) Find an expression for $y = f^{-1}(x)$, the inverse of $y = f(x)$. **2**

(iii) State the DOMAIN and RANGE of $y = f^{-1}(x)$. **2**

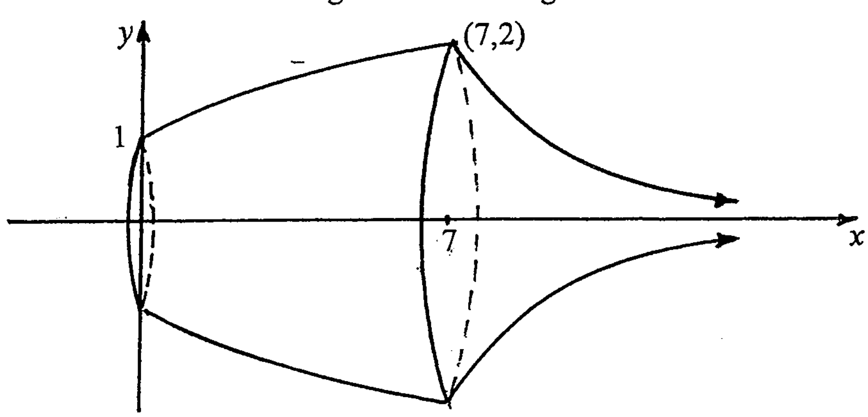
(iv) On the same graph, draw a neat sketch of $y = f(x)$ and $y = f^{-1}(x)$, clearly identifying each sketch. **1**

QUESTION 7 (9 marks)

(a) (i) Differentiate $y = \dfrac{x}{1+x^2}$. **2**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^2}$. **2**

(b) The shape below is made from the area between the x,y - axes, the line $x = 7$ and the curve $y = (x+1)^{\frac{1}{3}}$ joined with the area between the $x-$axis, the line $x = 7$, the curve $y = \dfrac{12}{x-1}$ and the line $x = K$ for $K > 7$ which are rotated one revolution about the $x-$axis. Find the limiting volume of the generated solid. **5**



END of PAPER

YEAR 11 TERM 4 2007

EXTENSION 1 SOLUTIONS

1 (a) $\cos \frac{7\pi}{6}$

$= -\cos \frac{\pi}{6}$

$= -\frac{\sqrt{3}}{2}$ (1)

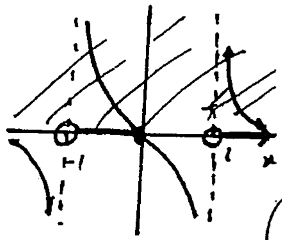
(b) $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$

$= \left[\sin^{-1}\frac{x}{3}\right]_0^{\frac{3}{2}}$ 1

$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$ 1

$= \frac{\pi}{6}$ 1 (3)

(c) $\frac{x}{x^2-1} \geq 0$



1 1 1 (3)

$\therefore -1 < x \leq 0$ or $x > 1$

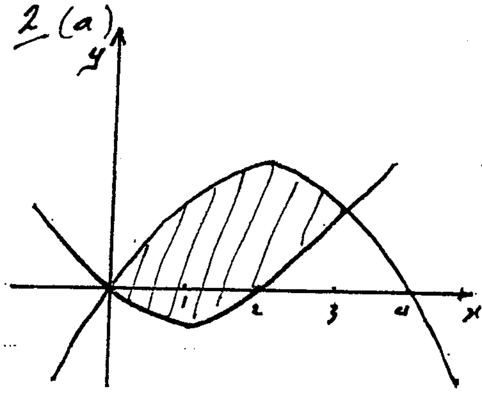
(d) $I = \frac{1}{3}\left[\frac{\ln 1}{1} + 4\frac{\ln 2}{2} + 2\frac{\ln 3}{3} + 4\frac{\ln 4}{4} + \frac{\ln 5}{5}\right]$ 1

$= \frac{1}{3}\left(0 + 2\ln 2 + 2\frac{\ln 3}{3} + \ln 4 + \frac{\ln 5}{5}\right)$

$= 1.3$ (to 1 dec. pl.) 1 (2)

/9

2 (a)



$4x - x^2 = x^2 - 2x$

$\therefore 2x^2 - 6x = 0$

$\therefore 2x(x-3) = 0$ 1

$\therefore x = 0, 3$

$\therefore$ Area $= \int_0^3 [(4x - x^2) - (x^2 - 2x)]\,dx$ 1

$= \int_0^3 (6x - 2x^2)\,dx$

$= \left[3x^2 - \frac{2x^3}{3}\right]_0^3$ (3)

$= 27 - 18$

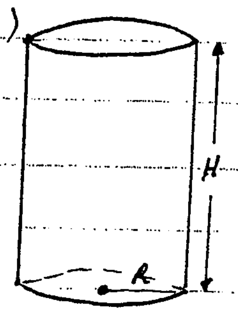
$= 9\,u^2$ 1

(b) $y = \frac{1}{\cos^{-1} x} = (\cos^{-1} x)^{-1}$

$\therefore \frac{dy}{dx} = -(\cos^{-1} x)^{-2} \times -\frac{1}{\sqrt{1-x^2}}$ 1 1 (2)

$= \frac{1}{(\cos^{-1} x)^2 \sqrt{1-x^2}}$

(c)



$V = \pi R^2 H$

$A = 2\pi R(R+H) = 2\pi N$

$\therefore R^2 + RH = N$

$\therefore H = \frac{N - R^2}{R}$

$V = \pi R^2 H$

$= \pi R^2 \frac{(N - R^2)}{R}$

$\therefore V = \pi R N - \pi R^3$ 1

For max volume, $\frac{dV}{dR} = 0$

$\frac{dV}{dR} = \pi N - 3\pi R^2 = 0$

$\therefore R^2 = \frac{N}{3}$ 1

$\therefore R = \sqrt{\frac{N}{3}}$ for $R > 0$

Test: $\frac{d^2V}{dR^2} = -6\pi R$

$= -6\pi\sqrt{\frac{N}{3}}$ 1

< 0 concave down

$\therefore$ MAXIMUM for $R > 0$

When $R = \sqrt{\frac{N}{3}}$

$H = \frac{N - \left(\sqrt{\frac{N}{3}}\right)^2}{\sqrt{\frac{N}{3}}}$

$= \frac{2N}{3} \div \sqrt{\frac{N}{3}}$

$= \frac{2N}{3}\sqrt{\frac{3}{N}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \cdot \frac{\sqrt{N}}{\sqrt{N}}$

$= 2\sqrt{\frac{N}{3}}$ 1

$\therefore H = 2R$ (4)

/9

3.(a)

$I = \int_3^4 \frac{2x}{\sqrt{25-x^2}}\,dx$ $\quad u = 25 - x^2$

$\therefore du = -2x\,dx$

$\therefore I = -\int_{16}^{9} \frac{du}{\sqrt{u}}$ $\quad$ When $x=3$, $u=16$ | When $x=4$, $u=9$

$= \int_9^{16} u^{-\frac{1}{2}}\,du$

$= 2\left[u^{\frac{1}{2}}\right]_9^{16}$ |

$= 2\left(16^{\frac{1}{2}} - 9^{\frac{1}{2}}\right)$

$\therefore I = 2$ | $\quad$ 3

(b)(i) $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$

$\therefore \sqrt{3}\cos x - \sin x = R(\cos x \cos\alpha - \sin x \sin\alpha)$

$\therefore R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$

$\therefore \frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$ $\quad \therefore \tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \frac{\pi}{6}$ |

Now $R\sin\alpha = \frac{R}{2} = 1$ $\quad$ 2

$\therefore R = 2$ |

$\therefore \sqrt{3}\cos\ - \sin x = 2\cos\left(x + \frac{\pi}{6}\right)$

(ii) $2\cos\left(x+\frac{\pi}{6}\right)$ is maximum

when $2\cos\left(x+\frac{\pi}{6}\right) = 2$

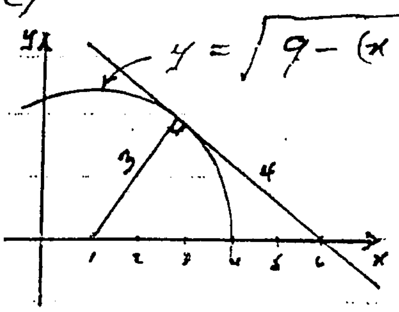
$\therefore \cos\left(x + \frac{\pi}{6}\right) = 1 = \cos 0 = \cos 2\pi = \ldots$

$\therefore x + \frac{\pi}{6} = 0, 2\pi, 4\pi, \ldots$

$\therefore x = -\frac{\pi}{6}, \frac{11\pi}{6}, \ldots$

$\therefore$ the first positive value of x is $\frac{11\pi}{6}$. $\quad$ 1

c)



Let equation be

$y - y_0 = m(x - x_0)$

Now $m = -\frac{3}{4}$

and (x_0, y_0) is $(6, 0)$

$\therefore y = -\frac{3}{4}(x-6)$

$\therefore 3x + 4y - 18 = 0$

4.

(a)(i) $y = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)$

$\therefore \frac{dy}{dx} = \frac{\cancel{2}}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{\cancel{2}\sqrt{x}} - \frac{1}{\sqrt{1-(2x-1)^2}} \times 2$ |

$= \frac{1}{\sqrt{1-x}\ \sqrt{x}} - \frac{2}{\sqrt{1-(4x^2-4x+1)}}$

$= \frac{1}{\sqrt{x(1-x)}} - \frac{2}{\sqrt{4x-4x^2}}$

$= \frac{1}{\sqrt{x(1-x)}} - \frac{\cancel{2}}{\cancel{2}\sqrt{x(1-x)}}$

$= 0$ | $\quad$ 2

(ii) Since $\frac{dy}{dx} = 0$, $\therefore y = f(x)$ is a constant.

Let $x = \frac{1}{4}$

$y = 2\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

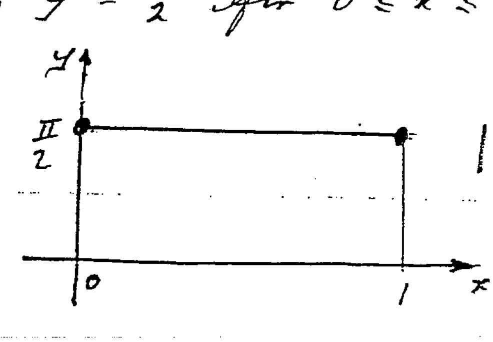
$= 3\sin^{-1}\frac{1}{2}$

$= 3 \cdot \frac{\pi}{6}$

$\therefore y = \frac{\pi}{2}$. | But $\sqrt{x}$ has domain $x \geq 0$ and $-1 \leq (2x-1) \leq 1$

$\therefore 0 \leq x \leq 1$.

$\therefore y = \frac{\pi}{2}$ for $0 \leq x \leq \frac{\pi}{2}$



| $\quad$ 2

4(b) When $x = 0$, $y = 3\cos^{-1} 0 = \frac{3\pi}{2}$

Equation of tangent is

$$y - \frac{3\pi}{2} = m(x - 0)$$

Now $m = \frac{dy}{dx}$ at $x = 0$

$$\frac{dy}{dx} = 3 \times -\frac{1}{\sqrt{1 - \frac{x^2}{16}}} \times \frac{1}{4}$$

$$= -\frac{3}{\sqrt{16 - x^2}} \quad 1$$

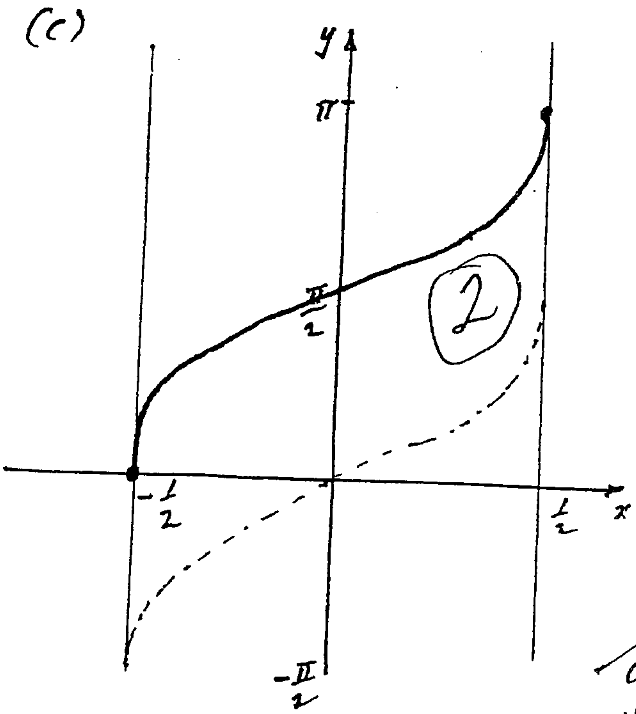
At $x = 0$, $m = -\frac{3}{4}$ 1

$\therefore y - \frac{3\pi}{2} = -\frac{3x}{4}$

$\therefore 3x + 4y - 6\pi = 0$ 1

3

(c)



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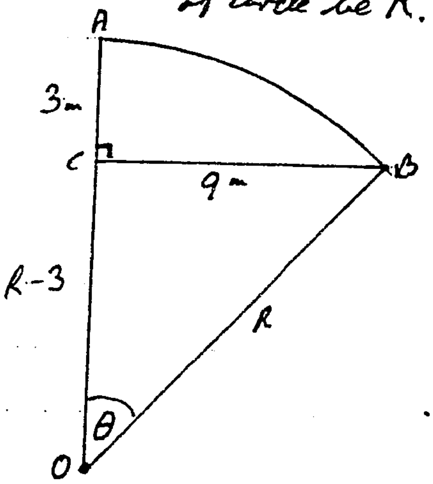
5(a)(i) Let the radius of circle be R.



$R^2 = (R-3)^2 + 9^2$

$\therefore R^2 = R^2 - 6R + 9 + 81$

$\therefore 6R = 90$

$\therefore R = 15$ 1

Let $A\hat{O}B = \theta$

$AB = R\theta$

$= 15\left(\sin^{-1}\frac{9}{15}\right)^c$ 1

$\therefore AB = 9.65$ cm. 1

3

(ii) Area $= \frac{1}{2}R^2\theta - \frac{1}{2}\cdot 12 \cdot 9$

$= \frac{1}{2} \times 15^2 \times 0.643 - 54$ 1

$= 18.34$ cm^2 1

2

18.39

(b)(i) Consider

$(a - b)^2 \geq 0$ for $a, b \in \mathbb{R}$

$\therefore a^2 + b^2 - 2ab \geq 0$

$\therefore a^2 + b^2 \geq 2ab$

1

(ii) Now $a + b \geq 2\sqrt{ab}$ $a > 0, b > 0$

also $\frac{1}{a} + \frac{1}{b} \geq 2 \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{b}}$ $a > 0, b > 0$ 1

$\therefore (a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 2\sqrt{ab} \cdot \frac{2}{\sqrt{ab}}$ $a > 0$, $b > 0$ 1

$\therefore (a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$ 1

3

OR Consider

$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) - 4$

$= (a + b)\frac{(a + b)}{ab} - 4$

$= \frac{(a + b)^2 - 4ab}{ab}$

$= \frac{a^2 + 2ab + b^2 - 4ab}{ab}$

$= \frac{(a - b)^2}{ab} \geq 0$ for $a > 0, b > 0$

$\therefore (a + b)\left(\frac{1}{a} + \frac{1}{b}\right) - 4 \geq 0$

$\therefore (a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$

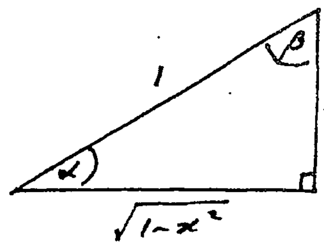
6 (a) Let $\alpha = \sin^{-1} x$ and $\beta = \cos^{-1} x$

L.H.S. $= \sin(2\sin^{-1} x + \cos^{-1} x)$ R.H.S. $= \sqrt{1-x^2}$

$= \sin(2\alpha + \beta)$

$= \sin 2\alpha \cos\beta + \cos 2\alpha \sin\beta$ 1

$= 2\sin\alpha\cos\alpha\cos\beta + (\cos^2\alpha - \sin^2\alpha)\sin\beta$



But $\sin\alpha = x$ and $\cos\beta = x$

$\therefore$ LHS $= 2 \cdot x \cdot \sqrt{1-x^2} \cdot x + [(1-x^2) - x^2]\sqrt{1-x^2}$ 1

$= 2x^2\sqrt{1-x^2} + \sqrt{1-x^2} - 2x^2\sqrt{1-x^2}$

$= \sqrt{1-x^2}$

$=$ RHS 1

3

(b)

(i) $y = \dfrac{\sqrt{x^2-1}}{x}$

$x^2 - 1 \geq 0$

$\therefore x^2 \geq 1$

$\therefore x \geq 1$ or $x \leq -1$ 1

$\therefore$ the largest POSITIVE domain is $x \geq 1$.

(ii) Interchange x and y.

$x = \dfrac{\sqrt{y^2-1}}{y}$

$\therefore xy = \sqrt{y^2-1}$

$\therefore x^2y^2 = y^2 - 1$

$\therefore y^2 - x^2y^2 = 1$

$\therefore y^2(1-x^2) = 1$

$\therefore y^2 = \dfrac{1}{1-x^2}$

$\therefore y = \pm\dfrac{1}{\sqrt{1-x^2}}$ 1

Since $y > 0$ for $x \geq 1$

$\therefore f^{-1}(x) = +\dfrac{1}{\sqrt{1-x^2}}$ 1

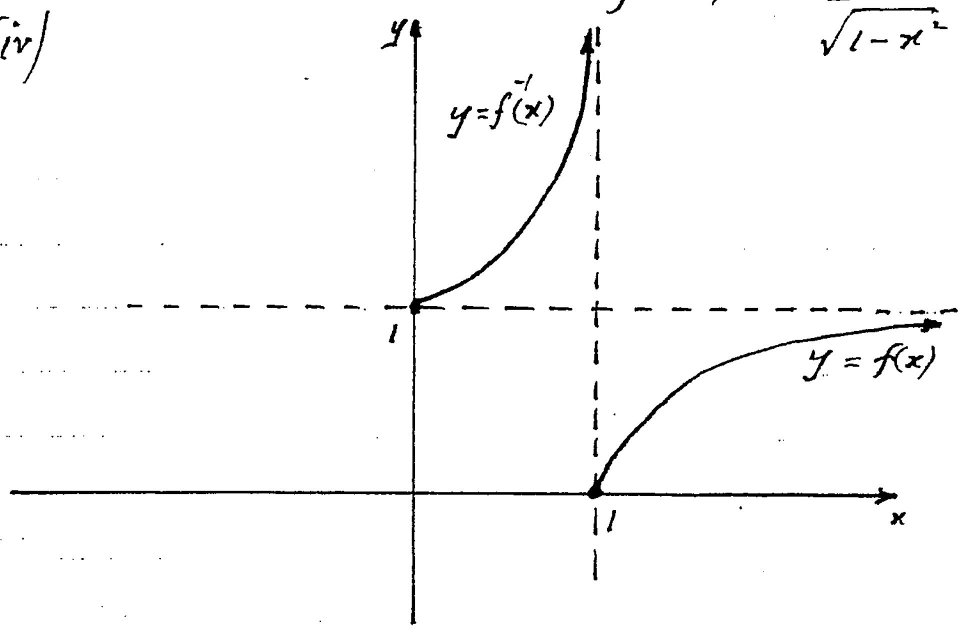
2

(iii) $D_f = \{x : 0 \leq x < 1\}$ 1

$R_f = \{y : y \geq 1\}$ 1

2

(iv)



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7 (a)

(i) $y = \dfrac{x}{1+x^2}$

$$\frac{dy}{dx} = \frac{(1+x^2).1 - x.2x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

2

(ii) $\dfrac{d}{dx}\left(\dfrac{x}{1+x^2}\right) = \dfrac{1-x^2}{(1+x^2)^2}$

$$= \frac{2-(1+x^2)}{(1+x^2)^2}$$

$$= \frac{2}{(1+x^2)^2} - \frac{1}{(1+x^2)}$$

$$\therefore \frac{1}{2}\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{1}{(1+x^2)^2} - \frac{1}{2(1+x^2)}$$

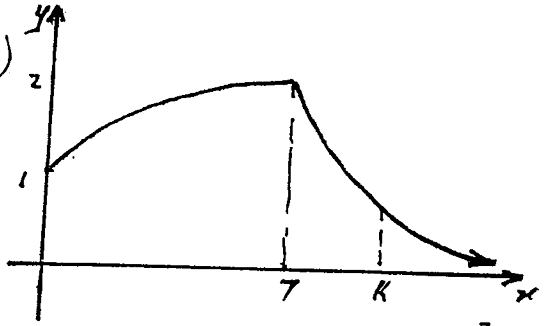
$$\therefore \int_0^1 \frac{dx}{(1+x^2)^2} = \frac{1}{2}\int_0^1 \frac{d}{dx}\left(\frac{x}{1+x^2}\right)dx + \frac{1}{2}\int_0^1 \frac{dx}{1+x^2}$$

$$= \frac{1}{2}\left(\frac{x}{1+x^2}\right)\Big]_0^1 + \frac{1}{2}\tan^{-1}x\Big]_0^1$$

$$\therefore \int_0^1 \frac{dx}{(1+x^2)^2} = \frac{1}{4} + \frac{\pi}{8}$$

2

(b)



$$\text{Volume} = \pi\int_0^7 (x+1)^{\frac{2}{3}}dx + \pi\int_7^K \frac{12^2}{(x-1)^2}dx$$

$$= \pi\left[\frac{3}{5}(x+1)^{\frac{5}{3}}\right]_0^7 + 144\pi\int_7^K (x-1)^{-2}dx$$

$$= \frac{3\pi}{5}\left(8^{\frac{5}{3}} - 1\right) + 144\pi\left[-\frac{1}{(x-1)}\right]_7^K$$

$$= \frac{3\pi}{5}\left(2^5 - 1\right) + 144\pi\left(\frac{1}{6} - \frac{1}{K-1}\right)$$

As $K \to \infty$, $\dfrac{1}{K-1} \to 0$

5

$\therefore$ Limiting volume $= \dfrac{3\pi}{5}(31) + \dfrac{144\pi}{6}$

$$= \frac{93\pi}{5} + 24\pi$$

$$= \frac{213\pi}{5} \; u^3$$

9